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LETTER TO THE EDITOR

Integrable open spin chains with non-symmetric R-matrices

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Abstract. We extend Sklyanin's formalism for constructing integrable open spin chains to the case of 'non-symmetric' (PT-invariant) R-matrices. We use this formalism to show that the Hamiltonian of an integrable open chain is invariant under gauge transformations of the R-matrix.

The quantum inverse scattering method (QISM) [1–5] was first developed for systems with periodic boundary conditions. This method was later generalized to systems on a finite interval with independent boundary conditions on each end by Sklyanin [6], relying on previous results of Cherednik [7] and Zamolodchikov [8]. Sklyanin used this formalism to solve the open spin-$\frac{1}{2}$ $A_1^{(1)}$ quantum spin chain with boundary terms. (This model was first solved in [9] by the coordinate Bethe ansatz approach.) Recently, Sklyanin's formalism and the fusion procedure [4] have been used [10] to solve the corresponding spin-1 chain. Considerable attention has been focused on these open chains, since for a particular choice of boundary terms, the models have the quantum algebra symmetry $U_q[\mathrm{SU}(2)]$; and when q is a root of unity, the models are related [11] to rational conformal field theories.

Let us recall that the R-matrix is a matrix $R(u)$ acting in the tensor product space $C^n \otimes C^n$, which is a function of the complex variable u called the spectral parameter, and which obeys the so-called Yang–Baxter equation

$$R_{12}(u-v)\, R_{13}(u)\, R_{23}(v) = R_{23}(v)\, R_{13}(u)\, R_{12}(u-v)\,. \tag{1}$$

Here $R_{12}(u)$, $R_{13}(u)$, and $R_{23}(u)$ are matrices acting in $C^n \otimes C^n \otimes C^n$, with $R_{12}(u) = R(u) \otimes 1$, $R_{23}(u) = 1 \otimes R(u)$, etc.

Sklyanin's formalism assumes that the R-matrix is 'symmetric'; i.e., that R has both P and T symmetry

$$\mathcal{P}_{12}\, R_{12}(u)\, \mathcal{P}_{12} = R_{12}(u) \qquad R_{12}^{t_1 t_2}(u) = R_{12}(u)\,.$$

Here $\mathcal{P}_{12}$ is the permutation matrix in $C^n \otimes C^n$,

$$\mathcal{P}_{12}(x \otimes y) = y \otimes x \qquad \text{for } x, y \in C^n \tag{2}$$

which has matrix elements

$$_{\alpha\beta}\left(\mathcal{P}_{12}\right)_{\alpha'\beta'} = \delta_{\alpha\beta'}\, \delta_{\beta\alpha'}\,.$$

Moreover, t_i denotes transposition in the ith vector space.

In this letter, we extend Sklyanin's formalism to the case of a 'non-symmetric' R-matrix, which satisfies the less restrictive condition of PT symmetry

$$\mathcal{P}_{12}\, R_{12}(u)\, \mathcal{P}_{12} = R_{12}^{t_1 t_2}(u)\,. \tag{3}$$

All of the R-matrices listed by Bazhanov [12] and Jimbo [13] are of this type. In [14], we use this formalism to show that a $U_q[\mathrm{SU}(2)]$-invariant spin chain constructed from the $A_2^{(2)}$ R-matrix is integrable. As a further application of this formalism, we show below that the Hamiltonian of an integrable open chain is invariant under 'gauge' (or 'symmetry-breaking') transformations of the R-matrix.

In addition to the PT symmetry property (3), we assume also that the R-matrix satisfies the unitarity condition

$$R_{12}(u)\, R_{12}^{t_1 t_2}(-u) = \zeta(u) \tag{4}$$

where $\zeta(u)$ is some even scalar function, as well as the crossing-unitarity condition

$$R_{12}(u) = \overset{1}{V}\, R_{12}^{t_2}(-u-\rho)\, \overset{1}{V}{}^{-1}\,. \tag{5}$$

Following Sklyanin, we use the notation

$$\overset{1}{V} \equiv V \otimes 1 \qquad \overset{2}{V} \equiv 1 \otimes V\,. \tag{6}$$

Bazhanov shows [12] that a large class of trigonometric R-matrices possesses crossing symmetry, and he gives a formula for the corresponding V matrices. (There are some R-matrices, such as $A_n^{(1)}$ for $n > 1$, which do not have crossing symmetry. The corresponding spin chains cannot be treated by the formalism presented here.) The unitarity and crossing-unitarity conditions imply that

$$R_{12}^{t_1}(u)\, \overset{1}{M}\, R_{12}^{t_2}(-u-2\rho)\, \overset{1}{M}{}^{-1} = \zeta(u+\rho) \tag{7}$$

where the matrix M is given by

$$M \equiv V^t\, V = M^t\,. \tag{8}$$

The primary ingredients for constructing an integrable open chain are the reflection matrices $K_-(u), K_+(u)$, which satisfy the following generalized Sklyanin relations:

$$R_{12}(u-v)\, \overset{1}{K}_-(u)\, R_{12}^{t_1 t_2}(u+v)\, \overset{2}{K}_-(v) = \overset{2}{K}_-(v)\, R_{12}(u+v)\, \overset{1}{K}_-(u)\, R_{12}^{t_1 t_2}(u-v) \tag{9}$$

and

$$\begin{aligned} R_{12}(-u+v)\, \overset{1}{K}{}_+^{t_1}(u)\, \overset{1}{M}{}^{-1}\, R_{12}^{t_1 t_2}(-u-v-2\rho)\, \overset{1}{M}\, \overset{2}{K}{}_+^{t_2}(v) \\ = \overset{2}{K}{}_+^{t_2}(v)\, \overset{1}{M}\, R_{12}(-u-v-2\rho) \overset{1}{M}{}^{-1}\, \overset{1}{K}{}_+^{t_1}(u)\, R_{12}^{t_1 t_2}(-u+v) \end{aligned} \tag{10}$$

respectively. The normalization $K_-(0) = 1$ is assumed. As in the case considered by Sklyanin, these relations are suggested by the compatibility of reflection by a wall with factorizable scattering. (For a detailed exposition, see the last paper in [10].)

As usual, the monodromy matrix $T(u)$ is given by

$$T(u) = T_+(u)\, T_-(u) \tag{11}$$

where the matrices $T_\mp(u)$ are given by

$$\begin{aligned} T_-(u) &\equiv L_n(u) L_{n-1}(u) \cdots L_1(u) \\ T_+(u) &\equiv L_N(u) L_{N-1}(u) \cdots L_{n+1}(u) \end{aligned} \tag{12}$$

and n is any integer between 1 and N. These matrices satisfy the fundamental relation

$$R_{12}(u-v)\, \overset{1}{T}_\mp(u)\, \overset{2}{T}_\mp(v) = \overset{2}{T}_\mp(v)\, \overset{1}{T}_\mp(u)\, R_{12}(u-v)\,. \tag{13}$$

Using (9) and (13), one can show that the quantity $\mathcal{T}_-(u)$ defined as

$$\mathcal{T}_-(u) \equiv T_-(u)\, K_-(u)\, T_-^{-1}(-u) \tag{14}$$

satisfies the same relation as $K_-(u)$

$$R_{12}(u-v)\, \overset{1}{\mathcal{T}}_-(u)\, R_{12}^{t_1 t_2}(u+v)\, \overset{2}{\mathcal{T}}_-(v) = \overset{2}{\mathcal{T}}_-(v)\, R_{12}(u+v)\, \overset{1}{\mathcal{T}}_-(u)\, R_{12}^{t_1 t_2}(u-v)\,. \tag{15}$$

Moreover, let us assume that the monodromy matrix satisfies the crossing-unitarity property

$$\left\{T^a(u)\right\}^a = \theta(u)\, M\, T(u-2\rho)\, M^{-1} \tag{16}$$

where $\theta(u)$ is some scalar function, and a denotes the antipode

$$T^a(u) \equiv \left\{T^{-1}(u)\right\}^t\,. \tag{17}$$

Using also the fact that

$$\overset{1}{M}{}^{-1}\, R_{12}(u)\, \overset{1}{M} = \overset{2}{M}\, R_{12}(u)\, \overset{2}{M}{}^{-1} \tag{18}$$

one can now show that the quantity $\mathcal{T}_+(u)$ given by

$$\mathcal{T}_+^t(u) \equiv T_+^t(u)\, K_+^t(u)\, T_+^a(-u) \tag{19}$$

satisfies the same relation as $K_+(u)$

$$\begin{aligned} &R_{12}(-u+v)\, \overset{1}{\mathcal{T}}{}_+^{t_1}(u)\, \overset{1}{M}{}^{-1}\, R_{12}^{t_1 t_2}(-u-v-2\rho)\, \overset{1}{M}\, \overset{2}{\mathcal{T}}{}_+^{t_2}(v) \\ &\quad = \overset{2}{\mathcal{T}}{}_+^{t_2}(v)\, \overset{1}{M}\, R_{12}(-u-v-2\rho)\, \overset{1}{M}{}^{-1}\, \overset{1}{\mathcal{T}}{}_+^{t_1}(u)\, R_{12}^{t_1 t_2}(-u+v)\,. \end{aligned} \tag{20}$$

We remark that there exists the following useful isomorphism: given a solution $\mathcal{T}_-(u)$ of (15), the quantity

$$\mathcal{T}_+(u) = \mathcal{T}_-^t(-u-\rho)\,M \tag{21}$$

satisfies (20). The proof proceeds by substituting into (20), and making use of the property (18).

By a suitable generalization of Sklyanin's arguments, it now follows that the transfer matrix $t(u)$, which is given by

$$t(u) \equiv \mathcal{T}_+(u)\,\mathcal{T}_-(u) = \operatorname{tr} K_+(u)\,T(u)\,K_-(u)\,T^{-1}(-u) \tag{22}$$

constitutes a one-parameter commutative family

$$[t(u), t(v)] = 0\,. \tag{23}$$

A monodromy matrix $T(u)$ can be constructed by taking

$$L_n(u) = R_{0n}(u) \qquad n = 1, 2, \ldots, N \tag{24}$$

in (12). Indeed, for this choice, the fundamental relation (13) and the crossing-unitarity condition (16) are satisfied. Provided that the R-matrix is regular (i.e., $R(0) = \mathcal{P}$), it follows that the integrable Hamiltonian H is given by

$$H = \frac{1}{2\operatorname{tr} K_+(0)}\left[t'(0) - \operatorname{tr} K_+'(0)\right] \tag{25}$$

$$= \sum_{n=1}^{N-1} H_{n,n+1} + \tfrac{1}{2}\overset{1}{K}{}_-'(0) + \frac{\operatorname{tr}_0 \overset{0}{K}_+(0) H_{N,0}}{\operatorname{tr} K_+(0)} \tag{26}$$

where the two-site Hamiltonian is given by

$$H_{n,n+1} = \mathcal{P}_{n,n+1}\,R_{n,n+1}'(0)\,. \tag{27}$$

For a non-symmetric R-matrix, the latter quantity is *not* equal to $R_{n,n+1}'(0)\,\mathcal{P}_{n,n+1}$.

This concludes our extension of Sklyanin's formalism to the case of a PT-invariant R-matrix. (We do not discuss the algebraic Bethe ansatz here.) In [14], we make use of this formalism to show that an open $A_2^{(2)}$ chain which has the quantum algebra symmetry $U_q[\mathrm{SU}(2)]$ is completely integrable.

As a further application of this formalism, we now show that the Hamiltonian of an integrable open chain is invariant under 'gauge' (or 'symmetry-breaking') transformations of the R-matrix. We consider the gauge transformation

$$R_{12}(u-v) \;\rightarrow\; \tilde{R}_{12}(u-v) \equiv \overset{1}{B}(u)\,\overset{2}{B}(v)\,R_{12}(u-v)\,\overset{1}{B}(-u)\,\overset{2}{B}(-v) \tag{28}$$

where $B(u)$ is a diagonal matrix with the properties

$$B(u)\,B(v) = B(u+v) \qquad B(0) = 1 \tag{29}$$

as well as

$$\overset{1}{B}(u)\,R_{12}(v)\,\overset{1}{B}(-u) = \overset{2}{B}(-u)\,R_{12}(v)\,\overset{2}{B}(u)\,. \tag{30}$$

It is easy to verify that the Yang–Baxter equation (1) transforms covariantly. Moreover, using the identities

$$\mathcal{P}_{12}\,\overset{1}{B}\,\mathcal{P}_{12} = \overset{2}{B} \qquad \mathcal{P}_{12}\,\overset{2}{B}\,\mathcal{P}_{12} = \overset{1}{B} \tag{31}$$

we see that the gauge transformation preserves the symmetry property (3) and the unitarity condition (4). The crossing-unitarity properties (5), (7) are also maintained, with

$$\tilde{V} = V\,B(\rho) \qquad \tilde{M} = \tilde{V}^t\tilde{V} = B(\rho)\,M\,B(\rho)\,. \tag{32}$$

The spectral-parameter-dependent transformation considered in [15] is a special case of the gauge transformation (28), where the matrix $B(u)$ is given by

$$\begin{aligned} &{}_{\alpha}B(u)_{\alpha'} = b_{\alpha}(u)\,\delta_{\alpha\alpha'} \\ &b_{\alpha}(u) = \mathrm{e}^{\mu[(n+1)-2\alpha]u} \end{aligned} \qquad \alpha, \alpha' = 1, 2, \ldots, n \tag{33}$$

that is

$$B(u) = \begin{pmatrix} \mathrm{e}^{\mu(n-1)u} & & & & \\ & \mathrm{e}^{\mu(n-3)u} & & & \\ & & \ddots & & \\ & & & \mathrm{e}^{-\mu(n-3)u} & \\ & & & & \mathrm{e}^{-\mu(n-1)u} \end{pmatrix}. \tag{34}$$

Here μ is an arbitrary parameter. The condition (30) is satisfied, provided that the R-matrix has the property

$${}_{\alpha\beta}R(u)_{\alpha'\beta'} = 0 \qquad \text{unless } \alpha + \beta = \alpha' + \beta'\,. \tag{35}$$

For this $B(u)$, the gauge transformation (28) becomes

$${}_{\alpha\beta}\tilde{R}(u)_{\alpha'\beta'} = \exp[\mu(\alpha' - \alpha + \beta - \beta')u]\,{}_{\alpha\beta}R(u)_{\alpha'\beta'} \tag{36}$$

which is precisely the transformation considered in [15]. For instance, for the symmetric spin-$\frac{1}{2}$ $A_1^{(1)}$ R-matrix

$$R(u) = \frac{1}{\sinh\eta}\begin{pmatrix} \sinh(u+\eta) & & & \\ & \sinh u & \sinh\eta & \\ & \sinh\eta & \sinh u & \\ & & & \sinh(u+\eta) \end{pmatrix} \tag{37}$$

this transformation yields the symmetry-broken R-matrix

$$\tilde{R}(u) = \frac{1}{\sinh\eta}\begin{pmatrix} \sinh(u+\eta) & & & \\ & \sinh u & e^{2\mu u}\sinh\eta & \\ & e^{-2\mu u}\sinh\eta & \sinh u & \\ & & & \sinh(u+\eta) \end{pmatrix}. \tag{38}$$

It is known [15] that the Hamiltonian of an integrable periodic chain is invariant under gauge transformations. To demonstrate the corresponding result for an open chain, we first observe that the generalized Sklyanin relations (9) and (10) transform covariantly, provided that the K-matrices transform as follows:

$$\tilde{K}_-(u) = B(u)\,K_-(u)\,B(u) \qquad \tilde{K}_+^t(u) = B(-u)\,K_+^t(u)\,B(-u)\,. \tag{39}$$

Here we have used equation (32), as well as the assumption that $B(u)$ commutes with M

$$[B(u), M] = 0\,. \tag{40}$$

(In the case that M is a diagonal matrix, this condition is satisfied trivially.) Using the transformation property

$$\tilde{T}(u) = B(u)\,T(u)\,B(-u) \tag{41}$$

we further see that T_- and T_+^t transform in the same way as K_- and K_+^t

$$\tilde{T}_-(u) = B(u)\,T_-(u)\,B(u) \qquad \tilde{T}_+^t(u) = B(-u)\,T_+^t(u)\,B(-u)\,. \tag{42}$$

It is now evident that the transfer matrix $t(u)$ given by (22) is gauge invariant. Since the open chain Hamiltonian H given by (25) is essentially $t'(0)$, we conclude that H is also gauge invariant, up to the addition of an irrelevant constant.

It is instructive to check the invariance of the Hamiltonian directly from the formula (26). The point is that the gauge-transformed two-site Hamiltonian $\tilde{H}_{n,n+1}$ generates boundary terms

$$\tilde{H}_{n,n+1} = \mathcal{P}_{n,n+1}\,\tilde{R}'_{n,n+1}(0) = H_{n,n+1} + \overset{n+1}{B}{}'(0) - \overset{n}{B}{}'(0) \tag{43}$$

which are cancelled by additional contributions from the other terms in (26):

$$\begin{aligned} &\tfrac{1}{2}\overset{1}{\tilde{K}}{}'_-(0) = \tfrac{1}{2}\overset{1}{K}{}'_-(0) + \overset{1}{B}{}'(0) \\ &\frac{\mathrm{tr}_0\,\overset{0}{\tilde{K}}_+(0)\,\tilde{H}_{N,0}}{\mathrm{tr}\,\tilde{K}_+(0)} = \frac{\mathrm{tr}_0\,\overset{0}{K}_+(0)\,H_{N,0}}{\mathrm{tr}\,K_+(0)} - \overset{N}{B}{}'(0) + \text{constant} \end{aligned} \tag{44}$$

Finally, let us return to the example of the spin-$\frac{1}{2}$ $A_1^{(1)}$ chain. In the gauge that the R-matrix is symmetric (37), the reflection matrices which correspond to an $U_q[\mathrm{SU}(2)]$-invariant Hamiltonian are

$$K_-(u) = \begin{pmatrix} e^{-u} & \\ & e^{u} \end{pmatrix} \qquad K_+(u) = \begin{pmatrix} e^{u+\eta} & \\ & e^{-(u+\eta)} \end{pmatrix}\,. \tag{45}$$

Clearly, it is possible to make a gauge transformation so that

$$\tilde{K}_-(u) = 1 \qquad \tilde{K}_+(u) = \tilde{M} = \begin{pmatrix} e^{\eta} & \\ & e^{-\eta} \end{pmatrix} \tag{46}$$

and thus, the entire Hamiltonian (including boundary terms) is given by $\sum_{n=1}^{N-1} \mathcal{P}_{n,n+1} \tilde{R}'_{n,n+1}(0)$. The generalization of the latter result to spin chains constructed from other trigonometric R-matrices is discussed in [14].

In conclusion, we have generalized Sklyanin's formalism for constructing integrable open chains to the less restrictive case of R-matrices with PT invariance. We emphasize that, in general, an R-matrix cannot be brought to a symmetric form (i.e., separately P- and T-invariant) by a gauge transformation.

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